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# ON THE AVERAGE COVERING NUMBER OF PYRAMID AND CIRCULAR LADDER GRAPHS

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**Abstract.** There are many measures of reliability and vulnerability in graph theory. Covering is one of them, which is first mentioned by Beineke in 1969. A vertex and an edge are said to cover each other in a graph $G$, if they are incident in $G$. A vertex cover in $G$ is a set of vertices that covers all the edges of $G$. Dogan and Dundar have defined average covering number in 2013. The average covering numbers of helm, pyramid and circular ladder graphs are given in this paper.

**Keywords**: Graph theory, vulnerability, covering.

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## 1 Introduction

Let $G = (V(G), E(G))$ be a graph, where $V(G)$ and $E(G)$ are vertex and edge sets of $G$, respectively. A vertex and an edge are said to cover each other in a graph $G$, if they are incident in $G$. A vertex cover in $G$ is a set of vertices that covers all the edges of $G$. The minimum cardinality of a vertex cover in a graph $G$ is called the vertex covering number of $G$ and is denoted by $\beta(G)$.

In 2013, Dogan and Dundar have defined the average covering number, which is NP-hard computable. In a graph $G = (V(G), E(G))$, a subset $S_v \subseteq V(G)$ of vertices is called a cover set of $G$ with respect to $v$ or a local covering set of vertex $v$, if each edge of the graph is incident to at least one vertex of $S_v$. The local covering number with respect to $v$ is the minimum order of a cover that contains $v$ and it is the minimum cardinality among the $S_v$ sets, which is denoted by $\beta_v$. $\beta_v$-set shows the set which has the minimum order of a cover that contains $v$. The average covering number $\bar{\beta}(G)$ of a graph $G$ is

$$\frac{1}{|V(G)|}\sum_{v\in V(G)}\beta_v(G),$$

where $n \geq 2$ is order of $G$ and the sum is overall $n$ vertices. So, the average covering number is defined by the mean of local covering numbers Dogan & Dundar (2013).

We shall use the standard terminology of graph theory, as it is modified by Chartrand et al. (2016).

The Helm graph $H_n$ is the graph obtained from on $n+1$ wheel graph by joining a pendant edge at vertex of the cycle. $W_{n+1}$ is a graph that contains a cycle of vertex $n$ and for which every graph in the cycle is connected to one other graph vertex(which is known as the Hub). Let the hub vertex be $v_{n+1}$, vertices on the cycle be $v_1, v_2, ..., v_n$ and the end vertices of the graph be $u_1, u_2, ..., u_n$.

The join graph $C_n \vee N_k\,(n \geq 3, k \geq 1)$, where $N_k$ is the null graph of order $k$, is called a $k$-pyramid and is denoted by $kP\,(n)$. The 2-pyramid $C_n \vee N_2$ is called bipyramid and is denoted by$BP\,(n)$. The 1-pyramid $C_n \vee N_1$ is the wheel graph $W_n$.

The cartesian product of two graphs $G_1$ and $G_2$ is commonly denoted by $G_1 \square G_2$, has vertex set $V(G) = V(G_1) \times V(G_2)$ where two distinct vertices $(u,\, v)$ and $(x,\, y)$ of $G_1 \square G_2$ are adjacent if either (1) $u = x$ and $vy \in E\,(G_2)$ or (2) $v = y$ and $ux \in E(G_1)$. As expected, $G_1 \square G_2 \cong G_2 \square G_1$ for all graphs $G_1$ and $G_2$.

The circular ladder graph $CL_n$ is the graph Cartesian product $L_n \square K_2$, where $K_2$ is the complete graph on two nodes and $C_n$ is the cycle graph on $n$ nodes. The graph $CL_n$ consists of two cycles namely top cycle and bottom cycle.

Let $G$ be a graph. As a vertex of $G$ is said to be a pendant vertex if and only if it has degree one.

**Theorem 1.** *If $G$ is vertex transitive, then $\bar{\beta}\,(G) = \beta\,(G)$ (Dogan & Dundar, 2013).*

## 2 Main results

In this section we shall mention about average covering number of helm graph $H_n$, pyramid graph $P_{(a\ ,b)}$ and circular ladder $CL_n$.

**Theorem 2.** *Let $H_n$ be a Helm graph with $n \geq 4$, $n \in \mathbb{Z}$. Then, average covering number of $H_n$ is*

$$\bar{\beta}(H_n) = n + \frac{n+1}{2n+1}.$$

*Proof.* Let $V(H_n) = \{v_1, v_2, ..., v_{2n+1}\}$. $\{v_1, v_2, ..., v_n, v_{2n+1}\}$ be set of hub vertex of $W_n$ and pendant vertices of $H_n$ and $\{v_{n+1}, v_{n+2}, ..., v_{2n}\}$ be other vertices of $H_n$. The reader can see the labeling of $H_4$ in figure 1.

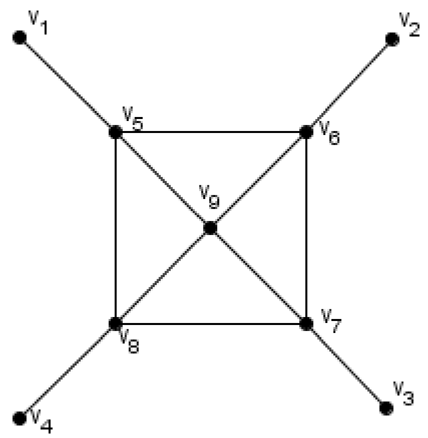


**Figure 1:** Labeling of $H_4$

There are two cases to obtain average covering number of $H_n$.

**Case 1:** If $v_i \in \{v_1, v_2, ..., v_n, v_{2n+1}\}$, then $\beta_{v_i} = |\{v_1, v_2, ..., v_n, v_{2n+1}\}| = n + 1$. This local covering set is not unique but the local covering numbers of all the sets are the same. Thus local covering number of any element in $\{v_1, v_2, ..., v_n, v_{2n+1}\}$ is $n + 1$. Therefore, $\beta_{v_i} \leq n + 1$.

Let $\{v_1, v_2, ..., v_n, v_{2n+1}\}$ be an average covering set of $H_n$. If any vertex of this set is removed from then it can not be an average covering set of $H_n$. Then, $\beta_{v_i} \geq n + 1$. Thus, $\beta_{v_i} = n + 1$.

**Case 2:** If $v_i \in \{v_{n+1}, v_{n+2}, ..., v_{2n}\}$, then $\beta_{v\,i} = |\{v_{n+1}, v_{n+2}, ..., v_{2n}\}| = n$. This local covering set is not unique but it is a $\beta_{v\,i}$-set. Thus local covering number of any element in $\{v_{n+1}, v_{n+2}, ..., v_{2n}\}$ is $n$. Therefore, $\beta_{v\,i} \leq n$. It is easy to see that $\beta_{v\,i} \geq n$.
Thus, $\beta_{v\,i} = n$.
From these two cases,

$$\bar{\beta}(H_n) = n + \frac{n+1}{2n+1}.$$

□

**Theorem 3.** *Let $P_{(a\ ,b)}$ be a pyramid graph with $a, b \in \mathbb{Z}$. The average covering number of $P_{(a\ ,b)}$ is*

$$\bar{\beta}\,(\,P_{(a\ ,b)}) \ = \begin{cases} a + \lceil \frac{b}{2} \rceil, & \frac{b}{2} \geq a \\ \frac{b^2+a(b+1)}{a+b}, & \frac{b}{2} < a \end{cases} .$$

*Proof.* $P_{(a\ ,b)} \ = aK_1 + C_b$. Let $V\left(P_{(a\ ,b)}\right) = \{v_1, v_2, ...., v_a, v_{a+1}, ..., v_{a+b}\}$ be the vertex set of $P_{(a\ ,b)}$. The reader can see labeling of $P_{(3,4)}$ in figure 2.

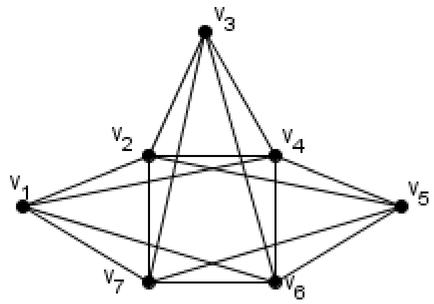


**Figure 2:** Labeling of $P_{(3,4)}$

There are two cases to obtain average covering number of the pyramid graph.

**Case 1:** Let $\frac{b}{2} \geq a$, $V\left(P_{(a\ ,b)}\right) = \{v_1\,, v_2\,, \ldots,\ v_a\,, v_{a+1}, v_{a+2}, ..., v_{a+b}\}$. In order to cover edges of $C_b$, $\lceil \frac{b}{2} \rceil$ vertices should be taken to local covering set and also for the rest of edges vertices $v_1, v_2, ...., v_a$ should be taken to the same set. Therefore, $\beta_v \leq a + \lceil \frac{b}{2} \rceil$ .

Let $\beta_v - set = \{\,v_1, v_2, ...., v_a\,, v_{a+1}\,,\ v_{a+3}, \ \ldots\,,\ v_{a+b}\}$ be an average covering set for $P_{(a\ ,b)}$. If any vertex of this set is removed then it is not an average covering set for $P_{(a\ ,b)}$. Therefore, $\beta_v \ \geq a + \lceil \frac{b}{2} \rceil$ .

Thus, $\bar{\beta}\,(\,P_{(a\ ,b)}) \ = a + \lceil \frac{b}{2} \rceil$ .

**Case 2:** Let $\frac{b}{2} \ < a$. There are two subcases.
**i)** If $v_i \in V\,(C_b)$, then $\beta_v \leq \ b$. It is easy to see that $\beta_v \geq \ b$. Thus, $\beta(\ P_{(a\ ,b)}) = b$.
**ii)** If $v_i \notin V\,(C_b)$, then $\beta_v \leq \ b+1$ and also $\beta_v \geq \ b+1$.
Thus, $\beta(\ P_{(a\ ,b)}) = b+1$.
Therefore, $\bar{\beta}(\ P_{(a\ ,b)}) = \ \frac{a(b+1)+b^2}{a+b}$.
From these two cases,

$$\bar{\beta}\,(\,P_{(a\ ,b)}) \ = \begin{cases} a + [\frac{b}{2}], & \frac{b}{2} \geq a \\ \\ \frac{b^2+a(b+1)}{a+b}, & \frac{b}{2} < a\ . \end{cases}$$

□

**Theorem 4.** *Let $CL_n$ be a circular ladder graph with $n \geq 2$ and $n \in \mathbb{Z}$. Then average covering number of $CL_n$ is*

$$\bar{\beta}(CL_n) = \begin{cases} n+1\,, & n \;\; odd \\ n & ,n \;\; even \end{cases} = 2\left\lceil \frac{n}{2} \right\rceil.$$

*Proof.* Let $V(CL_n) = \{v_{11}, v_{12}, v_{21}, v_{22}, ..., v_{n1}, v_{n2}\}$. The reader can see the labeling of $CL_4$ in figure 3. **Case 1:** Let n be even.

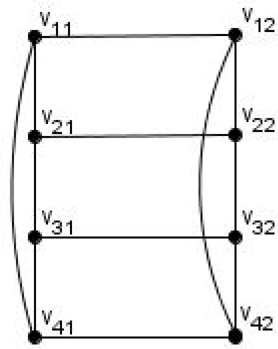


**Figure 3:** Labeling of $CL_4$

$$\beta_{v_{11}}(CL_n) = \left|\left\{v_{11}, v_{22}, v_{31}, v_{42}, ..., v_{(n-1)1}, v_{n2}\right\}\right| = n,$$

$$\beta_{v_{12}}(CL_n) = \left|\left\{v_{12}, v_{21}, v_{32}, v_{41}, ..., v_{(n-1)2}, v_{n1}\right\}\right| = n.$$

From theorem 2 $\beta_v \leq n$ and also $\beta_v \geq n$.
Therefore, $\bar{\beta}(CL_n) = n$.

**Case 2:** Let n be odd.

$$\beta_{v_{11}}(CL_n) = \left|\left\{v_{11}, v_{22}, v_{31}, v_{42}, ..., v_{(n-1)2}, v_{n1}, v_{n2}\right\}\right| = n+1,$$

$$\beta_{v_{12}}(CL_n) = \left|\left\{v_{12}, v_{21}, v_{32}, v_{41}, ..., v_{(n-1)1}, v_{n1}, v_{n2}\right\}\right| = n+1.$$

From theorem 2, $\beta_v \leq n+1$. As previous proofs $\beta_v \geq n+1$.
Therefore, $\bar{\beta}(CL_n) = n+1$.
Consequently,

$$\bar{\beta}(CL_n) = \begin{cases} n+1, & n \;\; odd \\ n, & n \;\; even \end{cases} = 2\left\lceil \frac{n}{2} \right\rceil.$$

□

# 3 Conclusion

In this paper, we mention about average covering number of Helm graph with $n \geq 4$, $n \in \mathbb{Z}$, pyramid graph with $a, b \in \mathbb{Z}$ and circular ladder graph with $n \geq 2$ and $n \in \mathbb{Z}$. Theorem 2 gives average covering number of Helm graph, Theorem 3 gives average covering number of the pyramid graph and Theorem 4 gives results for circular ladder graph with same parameter. The circular ladder graph is obtained by Cartesian product of a cycle of length at least three and an edge. It is connected, planar and Hamiltonian and also bipartite when $n$ is even. Planar graphs are really important: In real life problems, for example if we can keep away from crossing electricity, water, natural gas lines, it will be safer and easier to install.

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